The University of Alabama at Birmingham (UAB)
Department of Physics

PH 462/562 – Classical Mechanics II – Spring 2006

Central Force Problem: *Summary & Highlights*

General Form of a Central Force: $\vec{F} = F(r)\hat{e}_r$

(1) $\nabla \times \vec{F} = 0 \Rightarrow \vec{F}$ *is conservative* $\Rightarrow E = const$

(2) $\dfrac{d\vec{L}}{dt} = \dfrac{d}{dt}\left(\vec{r}\times\vec{p}\right) = 0 \Rightarrow \vec{L} = const$ *(motion takes place in plane)*

Hence, polar coordinate system very convenient: $r(t) = ?;\ \ \theta(t) = ?$

$L = mr^2\dot{\theta} = const$

Complementary Approaches to Solve the Central Force Problem

(1) Equations of Motion in Polar Coordinates

$$m\ddot{r} - mr\dot{\theta}^2 = F(r)$$
$$mr\ddot{\theta} + 2m\dot{r}\dot{\theta} = 0$$

(2) Conservation of Energy

$$E = T + V = \tfrac{1}{2}mv^2 + V(r) \Rightarrow \sqrt{\tfrac{2}{m}}\,t = \int_{r_0}^{r} \frac{dr}{\sqrt{E - V(r) - \frac{L^2}{2mr^2}}} \Rightarrow$$

$$\Rightarrow r(t);\ \ \theta(t) \quad \left[from\ \dot{\theta} = \tfrac{L}{mr^2}\right]$$

(3) Analysis based on Effective Potential Energy

$$U_{eff}(r) = V(r) + \frac{L^2}{2mr^2}$$

(4) Differential Equation for Orbit (i.e., trajectory in real space)

$$\frac{d^2u}{d\theta^2} = -u - \frac{m}{L^2u^2}F\left(\tfrac{1}{u}\right) \quad \Rightarrow \quad u(\theta) = \frac{1}{r(\theta)}$$

Assignment # 9   Due: Friday, April 28, 2006

1. A particle of mass *m* moves in three dimensions with a potential energy $V(r) = \frac{1}{2}kr^2$, where *r* is the radial coordinate of the spherical coordinate system, and *k* is a positive constant.
    a. Make a sketch showing *V*(*r*), the centrifugal potential energy $\frac{L^2}{2mr^2}$, and the effective potential energy $U_{eff}(r)$ *(Treat the angular momentum L as a known, fixed constant, that can be determined from the initial conditions).*
    b. Find the equilibrium distance r_0 (i.e., the distance at which the particle can circle the origin with constant radius).
    c. By making a Taylor expansion of $U_{eff}(r)$ about the equilibrium point r_0 and neglecting all terms in $(r - r_0)^3$ and higher, find the frequency of small oscillations about the circular orbit if the particles are disturbed a little from the separation r_0.
    d. Set up and solve the differential equation of the orbit and find the general form of the trajectories in three-dimensional space.